J. Fluid Mech. (1971), *vol.* 46, *part* 3, *pp.* 611-622
Printed in Great Britain

The solitary wave in water of variable depth. Part 2

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(Received 10 September 1970)

This paper examines the deformation of a solitary wave due to a slow variation of the bottom topography. Differential equations which determine the slow variation of the parameters of a solitary wave are derived by a certain averaging process applied to the exact inviscid equations. The equations for the parameters are solved when the bottom topography varies only in one direction, and when the wave evolves from a region of uniform depth. The variation of amplitude with depth is determined and compared with some recent experimental results.

1. Introduction

This paper is primarily concerned with the deformation of a solitary wave caused by a slow variation of the bottom topography. The solitary wave, on a constant still water depth h, is a permanent progressing wave form consisting of a single elevation above the undisturbed surface, whose amplitude a and effective length λ (e.g. the width when the free surface is one-tenth of its maximum) are such that a/h and h^2/λ^2 are comparable small quantities. It was first observed by Russell (1837), and established theoretically by Lavrent'ev (1943, 1947) and Friedrichs & Hyer (1954); earlier Boussinesq (1871, 1872) had constructed a theoretical solution to the lowest order in a/h. In a previous paper (Grimshaw 1970) the one-dimensional modulations formed on the Boussinesq solitary wave by a slow variation of the depth were considered. Using the Boussinesq equations an asymptotic solution was obtained which described a slowly varying solitary wave; the slow variations were described by a set of transport equations. These were solved in the case that the wave develops from a region where h is constant; it was found that the variation of the amplitude is determined by conservation of energy in the wave, and this causes a to vary as h^{-1}.

In this paper the previous results are extended to a finite value of a/h and to two-dimensional modulations. In § 3 an asymptotic analysis of the exact inviscid equations is presented; a small parameter β is introduced so that $|\nabla h/h|$ is $O(\beta)$, and β^{-1} is a length scale much greater than the length of the wave. An asymptotic expansion is used whose leading term is a modulated solitary wave; that is, the amplitude, and the other parameters associated with the wave, vary on a length scale of $O(\beta^{-1})$. Transport equations for these parameters are derived from conservation laws. In § 4 these transport equations are solved in the case when the wave develops from a region where h is constant; the variation of the amplitude is determined by conservation of energy in the wave. In § 2 the solitary wave is calculated up to the term of $O((a/h)^3)$, and, correct to this order, an expression for the variation of a/h with h is derived. The results are graphed and compared with some recent experimental results on the shoaling of solitary waves by Camfield & Street (1969). The theory and experiments would appear to be in reasonable agreement for small bottom slopes.

2. The solitary wave

It will be assumed that the flow is irrotational and that the fluid is inviscid, incompressible and of constant density. It may be anticipated that the Froude number will be close to critical, and so we choose a length scale h_0, a velocity scale $(gh_0)^{\frac{1}{2}}$ and a time scale $h_0(gh_0)^{-\frac{1}{2}}$, where h_0 is a typical value for the still-water depth. The equations of motion for the velocity potential $\phi(\mathbf{x}, z, t)$ are

$$\nabla^2\phi + \phi_{zz} = 0 \quad \text{for} \quad -h < z < \zeta, \tag{2.1}$$

$$\nabla\phi \,.\, \nabla h + \phi_z = 0 \quad \text{for} \quad z = -h, \tag{2.2}$$

$$\zeta_t + \nabla\phi \,.\, \nabla\zeta - \phi_z = 0 \quad \text{for} \quad z = \zeta, \tag{2.3}$$

$$\phi_t + \tfrac{1}{2}(|\nabla\phi|^2 + \phi_z^2) + \zeta = 0 \quad \text{for} \quad z = \zeta, \tag{2.4}$$

where $\mathbf{x} = (x, y)$ are the horizontal co-ordinates, $\nabla = (\partial/\partial x, \partial/\partial y)$ is the horizontal gradient operator, $z = \zeta(\mathbf{x}, t)$ is the free surface, and $z = -h(\mathbf{x})$ is the undisturbed depth. Equations (2.2) and (2.3) are kinematic boundary conditions and (2.4) is the condition that the pressure be constant on the free surface. The horizontal velocity component is $\mathbf{u}$ (equals $\nabla\phi$), and the vertical velocity component is w (equals ϕ_z).

In the remainder of this section it will be assumed that h is constant. We shall seek a solution of (2.1), (2.2), (2.3) and (2.4) for which ζ, $\mathbf{u}$, w are functions only of z, and the phase

$$\theta = \kappa(\boldsymbol{\nu} \,.\, \mathbf{x} - ct), \tag{2.5}$$

where κ (wave-number) and c (wave speed) are constants, and $\boldsymbol{\nu}$ is a constant horizontal unit vector; $\mathbf{c} = c\boldsymbol{\nu}$ is the wave velocity. Thus we seek a solution of the form

$$\zeta = B + E(\theta), \tag{2.6}$$

$$\mathbf{u} = \mathbf{A} + \mathbf{U}(\theta, z), \quad w = W(\theta, z), \tag{2.7}$$

where $\mathbf{A}$, B are constants representing the mean velocity and mean height respectively, and defined so that E, $\mathbf{U}$, W and all their derivatives vanish as $|\theta| \to \infty$ (we are anticipating that any such solution will be even in θ). The corresponding form for the potential ϕ, consistent with (2.4), is

$$\phi = \mathbf{A} \,.\, \mathbf{x} - Ct + F(\theta, z), \tag{2.8}$$

where $\mathbf{U} = \boldsymbol{\nu} U$, $U = \kappa F_\theta$ and $W = F_z$, and C is a constant, related to the Bernoulli constant. Substitution of (2.6) and (2.8) into (2.4), and letting $|\theta| \to \infty$ implies that

$$C = B + \tfrac{1}{2}|\mathbf{A}|^2; \tag{2.9}$$

further substitution of (2.6) and (2.8) into the equations of motion gives

$$\kappa^2 F_{\theta\theta} + F_{z^*z^*} = 0 \quad \text{for} \quad -h^* < z^* < E, \tag{2.10}$$

$$F_{z^*} = 0 \quad \text{for} \quad z^* = -h^*, \tag{2.11}$$

$$-\kappa c^* E_\theta + \kappa^2 E_\theta F_\theta - F_{z^*} = 0 \quad \text{for} \quad z^* = E, \tag{2.12}$$

$$-\kappa c^* F_\theta + \tfrac{1}{2}(\kappa^2 F_\theta^2 + F_z^2) + E = 0 \quad \text{for} \quad z^* = E, \tag{2.13}$$

where

$$c^* = c - \mathbf{A}.\mathbf{v}, \quad h^* = h + B, \quad z^* = z - B. \tag{2.14}$$

Without loss of generality, we may select the origin of θ at the crest of the wave, so that

$$E|_{\theta=0} = a, \quad E_\theta|_{\theta=0} = 0. \tag{2.15}$$

For small a, we use the shallow water expansion method of Friedrichs & Hyers, and put

$$\left.\begin{aligned} E(\theta) &= a\{e_0(\psi) + ae_1(\psi) + a^2 e_2(\psi) + \ldots\}, \\ F(\theta, z^*) &= a^{\frac{1}{2}}\{\mathscr{F}_0(\psi, z^*) + a\mathscr{F}_1(\psi, z^*) + a^2\mathscr{F}_2(\psi, z^*) + \ldots\}, \\ c^* &= c_0 + ac_1 + a^2 c_2 + a^3 c_3 + \ldots, \\ \kappa &= \kappa_0\{1 + a\kappa_1 + a^2\kappa_2 + \ldots\}, \\ \text{and} \qquad \kappa_0 \psi &= a^{\frac{1}{2}}\theta, \end{aligned}\right\} \tag{2.16}$$

where each e_i, $\partial\mathscr{F}_i/\partial\psi$, $\partial\mathscr{F}_i/\partial z^*$, $i = 0, 1, 2, \ldots$ vanish as $|\theta| \to \infty$. Friedrichs & Hyers (1954) proved that this expansion method does yield an existence proof for the solitary wave, and thereby demonstrate that it will at least provide an asymptotic description of the exact solution. It is customary to first look for a periodic solution of the type (2.16) (the cnoidal waves of Korteweg & de Vries), and then find the solitary wave as the infinite period limit. By this technique Laitone (1960) calculated the solitary wave up to $O(a^2)$, and Chappelear (1962) up to $O(a^3)$. It is of some interest that the solitary wave may be found directly, and we give an outline of the procedure below.

First, substitution of (2.16) into (2.10) and (2.11) gives

$$\mathscr{F}_0 = f_0(\psi), \quad \mathscr{F}_1 = f_1(\psi) - \tfrac{1}{2}(z^* + h^*)^2 f_0''(\psi), \text{ etc.} \tag{2.17}$$

The boundary conditions (2.12) and (2.13) then become, respectively,

$$\left.\begin{aligned} -a^{\frac{3}{2}}(c_0 + ac_1 + \ldots)(1 + a\kappa_1 + \ldots)(e_0' + ae_1' + \ldots) + a^{\frac{5}{2}} f_0' e_0', \\ -a^{\frac{3}{2}}(-h^* f_0'' + a\{-h^* f_1'' - 2\kappa_1 h^* f_0'' + \tfrac{1}{6}h^{*3} f_0^{\text{iv}} - e_0 f_0''\} + \ldots) = 0, \end{aligned}\right\} \tag{2.18}$$

$$\begin{aligned} &a(e_0 + ae_1 + \ldots) + \tfrac{1}{2}a^2 (f_0')^2 + \ldots \\ &\quad - a(c_0 + ac_1 + \ldots)(1 + a\kappa_1 + \ldots)(f_0' + a\{f_1' - \tfrac{1}{2}h^{*2} f_0'''\} + \ldots) = 0, \end{aligned} \tag{2.19}$$

where the primes denote differentiations with respect to ψ. To the lowest order in a, these equations give

$$-c_0 e_0' + h^* f_0'' = 0, \quad e_0 - c_0 f_0' = 0, \tag{2.20}$$

and so

$$c_0^2 = h^*, \quad e_0 = h^{*\frac{1}{2}} f_0'.$$

To the next lowest order in a, we have

$$\left.\begin{aligned} -c_0 e_1' + h^* f_1'' &= (c_1 + \kappa_1) e_0' - f_0' e_0' - 2\kappa_1 h^* f_0'' - e_0 f_0'' + \tfrac{1}{6} h^{*3} f_0^{\text{iv}}, \\ e_1 - c_0 f_1' &= -\tfrac{1}{2}(f_0')^2 + (c_1 + \kappa_1) f_0' - \tfrac{1}{2} c_0 h^{*2} f_0'''. \end{aligned}\right\} \tag{2.21}$$

If these are to be compatible, then

$$\tfrac{1}{3}h^{*3}e_0'' + \tfrac{3}{2}e_0^2 - 2c_1c_0e_0 = 0. \tag{2.22}$$

The solution of this which satisfies (2.15) and vanishes as $|\theta| \to \infty$ is

$$e_0 = \operatorname{sech}^2 p\psi, \quad p^2 = \tfrac{3}{4}h^{*-3}, \quad 2c_1c_0 = 1. \tag{2.23}$$

This is the Boussinesq solitary wave. Applying the boundary conditions (2.18) and (2.19) yields, at each stage, a pair of equations similar to (2.21), with e_1, and f_1' replaced by e_i and f_i', $i = 2, 3, \ldots$ respectively. The compatibility condition, at each stage, takes the form

$$\tfrac{1}{3}h^{*3}e_i'' + 3e_0e_i - e_i = c_0H_i \quad (i = 1, 2, \ldots), \tag{2.24}$$

where H_i is known in terms of e_0, c_{i+1} and κ_i. For example,

$$H_1 = e_0(2c_2 - \tfrac{19}{20}h^{*-\frac{3}{2}} - 2\kappa_1 h^{*-\frac{1}{2}}) + e_0^2(3h^{*-\frac{3}{2}} + 3\kappa_1 h^{*-\frac{1}{2}}) + e_0^3(-\tfrac{3}{2}h^{*-\frac{3}{2}}). \tag{2.25}$$

The homogeneous part of (2.24) has e_0' for a solution, and so (2.24) may be integrated once to give

$$\tfrac{1}{3}h^{*3}(e_i'e_0' - e_ie_0'') = -c_0\int_{\psi}^{\infty} e_0'H_i\,d\psi \quad (i = 1, 2, \ldots). \tag{2.26}$$

The application of the condition (2.15) implies that

$$\int_0^{\infty} e_0'H_i\,d\psi = 0 \quad (i = 1, 2, \ldots) \tag{2.27}$$

and this determines c_{i+1}. To keep the expansion well-ordered, we now impose the condition that e_i should vanish as $|\theta| \to \infty$ at least as fast as e_0; this implies that the coefficient of e_0 in the right-hand side of (2.25) must vanish, and this determines κ_i. Equation (2.26) may then be integrated to give e_i.

This procedure was carried out as far as the terms e_2 and $\mathscr{F}_2$, and gave results agreeing with those quoted by Chappelear (1962). We find that, if

$$\alpha = a/h^*, \tag{2.28}$$

then, omitting terms of $O(\alpha^4)$,

$$\begin{aligned}
E/h^* &= \alpha S^2 - \alpha^2\tfrac{3}{4}S^2T^2 + \alpha^3\{\tfrac{5}{8}S^2T^2 - \tfrac{101}{80}S^4T^2\},\\
U/h^{*\frac{1}{2}} &= \alpha S^2 + \alpha^2\{-\tfrac{3}{4}S^2 + S^2T^2 + (1 + z^*/h^*)^2(\tfrac{3}{4}S^2 - \tfrac{9}{4}S^2T^2)\}\\
&\quad + \alpha^3\{\tfrac{21}{40}S^2 - S^2T^2 - \tfrac{6}{5}S^4T^2 + (1 + z^*/h^*)^2(-\tfrac{9}{4}S^2 + \tfrac{15}{4}S^2T^2 + \tfrac{15}{2}S^4T^2)\\
&\quad + (1 + z^*/h^*)^4(\tfrac{3}{8}S^2 - \tfrac{45}{16}S^4T^2)\},\\
W/h^{*\frac{1}{2}} &= (3\alpha)^{\frac{1}{2}}(1 + z^*/h^*)S^2T[\alpha + \alpha^2\{-\tfrac{3}{8} - 2S^2 + (1 + z^*/h^*)^2(-\tfrac{1}{2} + \tfrac{3}{2}S^2)\}\\
&\quad + \alpha^3\{-\tfrac{49}{640} - \tfrac{17}{20}S^2 - \tfrac{18}{5}S^4 + (1 + z^*/h^*)^2(-\tfrac{13}{16} - \tfrac{25}{16}S^2 + \tfrac{15}{2}S^4)\\
&\quad + (1 + z^*/h^*)^4(-\tfrac{3}{40} + \tfrac{9}{8}S^2 - \tfrac{27}{16}S^4)\}],
\end{aligned} \tag{2.29}$$

where $S = \operatorname{sech} p\psi$, $T = \tanh p\psi$; also

$$\left.\begin{aligned}
c &= h^{*\frac{1}{2}}[1 + \tfrac{1}{2}\alpha - \tfrac{3}{20}\alpha^2 + \tfrac{3}{56}\alpha^3 + \ldots],\\
\kappa/\kappa_0 &= 1 - \tfrac{5}{8}\alpha + \tfrac{71}{128}\alpha^2 + \ldots,
\end{aligned}\right\} \tag{2.30}$$

so that $p\psi = h^{*-1}(\frac{3}{4}\alpha)^{\frac{1}{2}}(\kappa/\kappa_0)\,\theta\kappa^{-1}$. The formulae (2.29) may be used to estimate the maximum height of the solitary wave. If we adopt the criterion used by Stokes that α attains its maximum value when the fluid velocity of the crest equals the wave speed, so that the crest attains the shape of a 120° angle wedge, we find that $\alpha_{\max} = 1{\cdot}21$. As an alternative Laitone (1960) proposed the criterion that α attains its maximum when the vertical velocity component on the free surface vanishes (other than at the crest); this gives $\alpha_{\max} = 0{\cdot}61$. Byatt-Smith (1970) has shown that if the solitary wave crest is a 120° angle wedge then $\alpha_{\max} = \frac{1}{2}c^2h^{*-1}$; using (2.30) this gives $\alpha_{\max} = 0{\cdot}92$. These results must be regarded as unsatisfactory compared with the estimate $\alpha_{\max} = 0{\cdot}78$ obtained by McGowan (1894) from a theoretical study of the highest wave. Recently Byatt-Smith, in a numerical study of the highest wave, obtained $\alpha_{\max} = 0{\cdot}86$. Camfield & Street (1969) found experimentally that $\alpha_{\max} = 0{\cdot}73$.

In the next section it is found useful to define the quantities

$$\hat{E} = \int_{-\infty}^{\infty} E\,d\theta, \quad \text{and} \quad \hat{U} = \int_{-\infty}^{\infty} U\,d\theta. \tag{2.31}$$

$\kappa^{-1}\hat{E}$ is the mass (apart from a constant proportionality factor ρh_0 where ρ is the density of the fluid) carried forward by the wave; $\kappa^{-1}\hat{U}$ is easily shown to be independent of z. From (2.39) we have

$$\left.\begin{aligned} \hat{E} &= h^*(2\alpha - \tfrac{1}{2}\alpha^2 + \tfrac{2}{25}\alpha^3 + \ldots), \\ \hat{U} &= h^{*\frac{1}{2}}(2\alpha - \tfrac{5}{6}\alpha^2 + \tfrac{19}{300}\alpha^3 + \ldots). \end{aligned}\right\} \tag{2.32}$$

3. Modulations caused by a slowly varying depth

It will now be supposed that h is a function of x, but is *slowly varying* in the sense that h varies little over a distance comparable with the effective length of the wave. Thus we shall assume that $h = h(\mathbf{X})$ where

$$\mathbf{X} = \beta\mathbf{x}, \quad T = \beta t, \tag{3.1}$$

where β is a small parameter. In this section we shall find equations which govern the modulations to the solitary wave of § 2 caused by this slow variation of the depth. This will be achieved by assuming that there is an asymptotic solution of the exact equations (2.1) to (2.4), whose leading term may be represented by the solution of § 2, but the parameters $\mathbf{A}$, B, C, c and $\boldsymbol{\kappa}$ (equals $\kappa\boldsymbol{\nu}$) which determine that solution are now slowly varying and so functions of $\mathbf{X}$, T. Whitham (1965*a,b*) has considered problems of this type for periodic slowly varying wave trains governed by non-linear, dispersive equations. The procedures described in this section are closely related to the procedures developed by Whitham and other workers in this field.

Thus we are motivated to seek an asymptotic solution of the exact equations (2.1) to (2.4) of the form

$$\left.\begin{aligned} \zeta &= B(\mathbf{X},T) + E(\theta;\mathbf{X},T) + \beta\zeta_1(\theta;\mathbf{X},T) + O(\beta^2), \\ \mathbf{u} &= \mathbf{A}(\mathbf{X},T) + \mathbf{U}(\theta,z;\mathbf{X},T) + \beta\mathbf{u}_1(\theta,z;\mathbf{X},T) + O(\beta^2), \\ w &= W(\theta,z;\mathbf{X},T) + \beta w_1(\theta,z;\mathbf{X},T) + O(\beta^2). \end{aligned}\right\} \tag{3.2}$$

$\mathbf{A}, B$ are determined so that E, $\mathbf{U}$ and W, and all their derivatives with respect to θ vanish as $|\theta| \to \infty$. The phase θ is such that

$$\nabla_x \theta = \boldsymbol{\kappa}, \quad \theta_t = -\kappa c \tag{3.3}$$

and so $$\theta = \beta^{-1}\Theta(\mathbf{X}, T), \quad \text{where} \quad \boldsymbol{\kappa} = \nabla_{\mathbf{X}}\Theta, \; -\kappa c = \Theta_T. \tag{3.4}$$

θ is a *fast* variable, which has yet to be determined, and $\mathbf{X}, T$ are *slow* variables; (3.2) is a two-scale asymptotic expansion of a type familiar in the context of ordinary differential equations. Since derivatives with respect to θ and z are $O(1)$, while derivatives with respect to $\mathbf{X}$ and T are $O(\beta)$, it is clear that when (3.2) and the corresponding expression (3.7) for the potential ϕ (derived below), are sustituted into (2.1) to (2.4), the terms of $O(1)$ are just (2.10) to (2.14) which describe the solitary wave of § 2, except that $\mathbf{A}$, B, C, c and $\boldsymbol{\kappa}$ are now functions of $\mathbf{X}$, T. The transport equations which determine these parameters are found by applying the principle that the asymptotic expansion (3.2) is to be well ordered, i.e. $\beta\zeta_1$, $\beta\mathbf{u}_1$ and βw_1 are $O(\beta)$ with respect to $B+E$, $\mathbf{A}+\mathbf{U}$, and W respectively for all θ. Thus we shall assume that

$$\mathbf{A}_1^{\pm} = \lim_{\theta\to\pm\infty} \mathbf{u}_1, \quad B_1^{\pm} = \lim_{\theta\to\pm\infty} \zeta_1, \quad D_1^{\pm} = \lim_{\theta\to\pm\infty} w_1 \tag{3.5}$$

all exist (and are functions of $\mathbf{X}$, T and z), and all derivatives of $\mathbf{u}_1$, ζ_1 and w_1 with respect to θ vanish as $|\theta| \to \infty$. Then we define

$$\left.\begin{aligned} \mathbf{A}_1 &= \tfrac{1}{2}(\mathbf{A}_1^+ + \mathbf{A}_1^-), & [\mathbf{u}_1] &= (A_1^+ - A_1^-), & \mathbf{U}_1 &= \mathbf{u}_1 - \mathbf{A}_1,\\ B_1 &= \tfrac{1}{2}(B_1^+ + B_1^-), & [\zeta_1] &= (B_1^+ - B_1^-), & E_1 &= \zeta_1 - B_1,\\ D_1 &= \tfrac{1}{2}(D_1^+ + D_1^-), & [w_1] &= (D_1^+ - D_1^-), & W_1 &= w_1 - D_1. \end{aligned}\right\} \tag{3.6}$$

It may be noted that this notation differs slightly from that used in Grimshaw (1970).

Next we seek an asymptotic expansion for the potential ϕ such that $\mathbf{u} = \nabla\phi$, $w = \phi_z$. The existence of such a potential implies that $\mathbf{A}_1$, and $[\mathbf{u}_1]$ are independent of z, and so are functions of $\mathbf{X}$, T alone (and also implies that $\mathbf{A}$ is independent of z). We find that

$$\phi = \beta^{-1}\psi(\mathbf{X}, T) + F(\theta, z; \mathbf{X}, T) + \psi_1(\mathbf{X}, T) + \beta F_1(\theta, z; \mathbf{X}, T) + \ldots, \tag{3.7}$$

where the remaining terms are $O(\beta^2)$ if they involve θ, and $O(\beta)$ otherwise, and

$$\nabla_{\mathbf{X}}\psi = \mathbf{A}, \qquad \psi_T = -C, \tag{3.8}$$

$$\nabla_{\mathbf{X}}\psi_1 = \mathbf{A}_1, \quad \psi_{1T} = -C_1, \tag{3.9}$$

$$F = \int_0^{\infty} \kappa^{-1} U(\theta', z; \mathbf{X}, T)\, d\theta', \tag{3.10}$$

$$\boldsymbol{\kappa} F_{1\theta} = \mathbf{U}_1 - \nabla_{\mathbf{X}} F. \tag{3.11}$$

It follows that

$$\phi_t = -C - cU + \beta(-C_1 - c\boldsymbol{\nu}\,.\,\mathbf{U}_1 + F_T + c\boldsymbol{\nu}\,.\,\nabla_{\mathbf{X}} F) + O(\beta^2). \tag{3.12}$$

The substitution of (3.7) into (2.1) and (2.2) yields (2.10) and (2.11) as the terms of $O(1)$, and a further two equations for the terms of $O(\beta)$; these latter equations, when evaluated as $\theta \to \pm\infty$, imply that

$$[w_1] = 0, \tag{3.13}$$

and $$D_1 = -z\,\mathrm{div}_{\mathbf{X}}\,\mathbf{A} - \mathrm{div}_{\mathbf{X}}\,(h\mathbf{A}). \tag{3.14}$$

(It may be noted that (2.10) and (2.11) imply that W tends to zero as $|\theta| \to \infty$.) Again, substitution of (3.7) into the consistency relation

$$\frac{\partial}{\partial t}(\nabla\phi) - \nabla(\phi_t) = 0, \tag{3.15}$$

yields for the term of $O(\beta)$,

$$\mathbf{A}_T + \nabla_{\mathbf{X}} C + \kappa^{-1} U\{\boldsymbol{\kappa}_T + \nabla_{\mathbf{X}}(\kappa c)\} = 0, \tag{3.16}$$

and letting $|\theta| \to \infty$, we have

$$\mathbf{A}_T + \nabla_{\mathbf{X}} C = 0, \tag{3.17}$$

whence

$$\boldsymbol{\kappa}_T + \nabla_{\mathbf{X}}(\kappa c) = 0. \tag{3.18}$$

These two equations are just the consistency relations for ψ and Θ respectively and provide two transport equations. A third is (2.9). Two more are needed, and may be determined as follows.

The substitution of (3.2) and (3.12) into the boundary conditions (2.3) and (2.4) yield (2.12) and (2.13) as the terms of $O(1)$, and a further two equations for the terms of $O(\beta)$; these latter equations, when evaluated as $\theta \to \pm\infty$, imply that

$$C_1 = B_1 + \mathbf{A}.\mathbf{A}_1, \tag{3.19}$$

$$[E_1] = \mathbf{c}^*.[\mathbf{u}_1] - (\kappa^{-1}\hat{U})_T - c\boldsymbol{\nu}.\nabla_{\mathbf{X}}(\kappa^{-1}\hat{U}), \tag{3.20}$$

and

$$B_T + \nabla_{\mathbf{X}}.(h^*\mathbf{A}) = 0, \tag{3.21}$$

where

$$\mathbf{c}^* = c\boldsymbol{\nu} - \mathbf{A}, \quad c^* = \boldsymbol{\nu}.\mathbf{c}^*;$$

$\hat{U}$ is defined in (2.31) and is a function of $\mathbf{X}$, T alone. Equation (3.21) is a fourth transport equation, which together with (2.9) and (3.17) determines $\mathbf{A}$, B and C. The terms of $O(\beta)$ obtained from the substitution of (3.2) and (3.12) into (2.1) to (2.4) now provide a *linear* boundary-value problem for E_1, F_1. The fifth transport equation may presumably be found by subjecting the solution of this problem to the principle that E_1, $\mathbf{u}_1$ and w_1 are bounded functions of θ. This procedure was carried out in the simpler context of the Boussinesq equations by Grimshaw (1970). Here we shall adopt the alternative procedure of assuming that E_1, $\mathbf{u}_1$ and w_1 exist as bounded functions of θ, and seeking the transport equations directly from averaged conservation laws (the procedure is analogous to that used by Whitham (1965*a*) for slowly varying periodic wave trains).

The typical conservation law has the form

$$\partial p/\partial t + \nabla_{\mathbf{x}}.\mathbf{q} + \beta r = 0, \tag{3.22}$$

where p, $\mathbf{q}$ and r are functions of $\mathbf{x}$, t and r is proportional to a component of $\nabla_{\mathbf{X}} h$, its presence being due to the inhomogeneity of the medium. In the present context there are three such conservation laws, there being one for mass, momentum and energy respectively.

(i) Mass:

$$p = \zeta, \quad \mathbf{q} = \int_{-h}^{\zeta} \mathbf{u}\, dz, \quad r = 0. \tag{3.23}$$

(ii) Momentum:

$$p = \int_{-h}^{\zeta} \mathbf{u}\, dz, \quad \mathbf{q} = \int_{-h}^{\zeta} \left(\mathbf{u}\mathbf{u} + \frac{\pi}{\rho}\mathbf{I}\right) dz, \quad r = -(\pi/\rho)_{z=-h} \nabla_{\mathbf{X}} h. \tag{3.24}$$

(iii) Energy:

$$p = \int_{-h}^{\zeta} (\tfrac{1}{2}|\mathbf{u}|^2 + \tfrac{1}{2}w^2 + z)\, dz, \quad \mathbf{q} = \int_{-h}^{\zeta} \mathbf{u}(\tfrac{1}{2}|\mathbf{u}|^2 + \tfrac{1}{2}w^2 + z + \pi/\rho)\, dz, \quad r = 0, \quad (3.25)$$

where the pressure

$$\pi = \rho(-\phi_t - \tfrac{1}{2}|\mathbf{u}|^2 - \tfrac{1}{2}w^2 - z), \quad (3.26)$$

and ρ is the density. In (3.24) $\mathbf{q}$ is a dyadic and $\mathbf{I}$ is the unit dyadic. Since ζ, $\mathbf{u}$, w have expansions of the form (3.2) it follows that p, $\mathbf{q}$, r have similar expansions, e.g.

$$p = p_0(\theta; \mathbf{X}, T) + \beta p_1(\theta; \mathbf{X}, T) + O(\beta^2). \quad (3.27)$$

Then our hypotheses on the expansions of ζ, etc., imply that

$$P_i^{\pm} = \lim_{\theta \to \pm\infty} p_i \quad (i = 0, 1) \quad (3.28)$$

exist, and we define

$$\bar{P}_i = \tfrac{1}{2}(P_i^+ + P_i^-), \quad [p_i] = P_i^+ - P_i^- \quad (i = 0, 1). \quad (3.29)$$

Since p_0, etc., are even in θ, $[p_0]$, etc., vanish but $[p_1]$, etc., are in general not zero. Also we observe that

$$\bar{P}_i = \lim_{\gamma \to \infty} \frac{1}{2\gamma} \int_{-\gamma}^{\gamma} p_i\, d\theta \quad (i = 0, 1). \quad (3.30)$$

Next we define the reduced (or wave) quantity by

$$P = p_0 - \bar{P}_0, \quad (3.31)$$

and its wave average by

$$\hat{P} = \lim_{\gamma \to \infty} \int_{-\gamma}^{\gamma} P\, d\theta. \quad (3.32)$$

We now substitute the expansion (3.27) into (3.22) and equate to zero the term of $O(1)$, and the term of $O(\beta)$; these give respectively

$$-cP + \boldsymbol{\nu} . \mathbf{Q} = 0, \quad (3.33)$$

and

$$\bar{P}_{0T} + \nabla_{\mathbf{X}} . \bar{\mathbf{Q}}_0 + \bar{R}_0 + P_T + \nabla_{\mathbf{X}} . \mathbf{Q} + R - \kappa c p_{1\theta} + \boldsymbol{\kappa} . \mathbf{q}_{1\theta} = 0. \quad (3.34)$$

Equation (3.33) is satisfied identically by the solitary wave solution of § 2. Then we take the mean of (3.34), i.e. the averaging procedure defined by (3.30); this yields

$$\bar{P}_{0T} + \nabla_{\mathbf{X}} . \bar{Q}_0 + \bar{R}_0 = 0. \quad (3.35)$$

Next we take the wave average of (3.34), i.e. the averaging procedure defined by (3.32); this yields

$$\hat{P}_T + \nabla_{\mathbf{X}} . \hat{\mathbf{Q}} + \hat{R} - \kappa c[p_1] + \boldsymbol{\kappa} . [\mathbf{q}_1] = 0. \quad (3.36)$$

The equations (3.35) and (3.36) are transport equations. The set obtained from (3.35) using (3.23), (3.24) and (3.25) involve only $\mathbf{A}$, B, C and may be derived from (2.9), (3.16) and (3.21). Equation (3.36) may be put in the form

$$\left.\begin{aligned} \hat{P}_T + \nabla_{\mathbf{X}} . (\mathbf{c}\hat{P} + \hat{\mathbf{Q}}') + \hat{R} &= \kappa\{c[p_1] - [\boldsymbol{\nu} . \mathbf{q}_1]\}, \\ \text{where} \quad \mathbf{Q}' &= \mathbf{Q} - \boldsymbol{\nu}(\boldsymbol{\nu} . \mathbf{Q}). \end{aligned}\right\} \quad (3.37)$$

The equation (3.37) may now be applied to (3.23), (3.24) and (3.25) in turn.

(i) Mass:

$$\frac{\partial}{\partial T}\hat{E}+\nabla_{\mathbf{X}}.\{(\mathbf{c}+\mathbf{A}')\hat{E}\} = \kappa\{c^*[E_1]-h^*[\boldsymbol{\nu}.\mathbf{u}_1]\}, \tag{3.38}$$

where $\mathbf{A}'$ is the component of $\mathbf{A}$ normal to $\boldsymbol{\nu}$ (it may be noted that $\mathbf{c}+\mathbf{A}' = c^*\boldsymbol{\nu}+\mathbf{A}$). Equation (3.38), together with (3.20) may be regarded as determining $[E_1]$ and $[\boldsymbol{\nu}.\mathbf{u}_1]$, since it follows from (3.11) that

$$\mathbf{A}'.[\mathbf{u}_1] = \mathbf{A}'.\nabla_{\mathbf{X}}(\kappa^{-1}\hat{U}). \tag{3.39}$$

(ii) Momentum: The equations obtained from (3.24) will be omitted, as they may be derived from the other transport equations and will not be needed in the sequel.

(iii) Energy: The equation obtained from (3.25), after elimination of $[E_1]$ and $[\mathbf{u}_1]$ through (3.20), (3.38) and (3.39), and subsequent simplification using (2.9), (3.17), (3.18), (3.21) and (3.22) is

$$\frac{\partial\mathscr{E}}{\partial T}+\nabla_{\mathbf{X}}.(\mathbf{c}\mathscr{E})+\mathbf{A}.\frac{\partial}{\partial T}\{\boldsymbol{\nu}(c^*\hat{E}-h^*\hat{U})\}+\nabla_{\mathbf{X}}.\{\mathbf{A}c(c^*\hat{E}-h^*\hat{U})\} + (c^*\hat{U}-\hat{E})\nabla_{\mathbf{X}}.(h^*\mathbf{A}) = 0, \tag{3.40}$$

where

$$\mathscr{E} = \int_{-\infty}^{\infty}\left\{\int_{-h^*}^{E}(\tfrac{1}{2}U^2+\tfrac{1}{2}W^2)\,dz^*+\tfrac{1}{2}E^2\right\}d\theta. \tag{3.41}$$

$\kappa^{-1}\mathscr{E}$ is the energy carried by the wave. Equation (3.40) is the fifth transport equation.

4. Solution of the transport equations

The transport equations are (2.9), (3.17), (3.18), (3.21) and (3.40) and are displayed here for convenience:

$$C = B+\tfrac{1}{2}|\mathbf{A}|^2, \tag{4.1}$$

$$\mathbf{A}_T+\nabla_{\mathbf{X}}C = 0, \tag{4.2}$$

$$B_T+\nabla_{\mathbf{X}}.(h^*\mathbf{A}) = 0, \tag{4.3}$$

$$\boldsymbol{\kappa}_T+\nabla_{\mathbf{X}}(\kappa c) = 0, \tag{4.4}$$

$$\mathscr{E}_T+\nabla_{\mathbf{X}}.(\mathbf{c}\mathscr{E})+\mathbf{A}.\{\boldsymbol{\nu}(c^*\hat{E}-h^*\hat{U})\}_T+\nabla_{\mathbf{X}}.\{\mathbf{A}c(c^*\hat{E}-h^*\hat{U})\} + (c^*\hat{U}-\hat{E})\nabla_{\mathbf{X}}.(h^*\mathbf{A}) = 0. \tag{4.5}$$

The first three equations involve only $\mathbf{A}$, B, C; they are, not unexpectedly, just the shallow-water equations, and can, in principle, be solved. Thus $\mathbf{A}$, B can be regarded as known when solving (4.4), and (4.5). In particular if $\mathbf{A}$, B both vanish at $T = 0$ for all $\mathbf{X}$, then they vanish for all T. This situation may arise, for example, when the wave evolves from a region where h is constant (say $h = 1$ for $X \leqslant 0$) and the wave profile is initially exactly that of a solitary wave. Under these conditions the equations to be solved are just (4.4), and (4.5) with the last three terms omitted. Also it is clear that

$$\mathscr{E} = \kappa V, \tag{4.6}$$

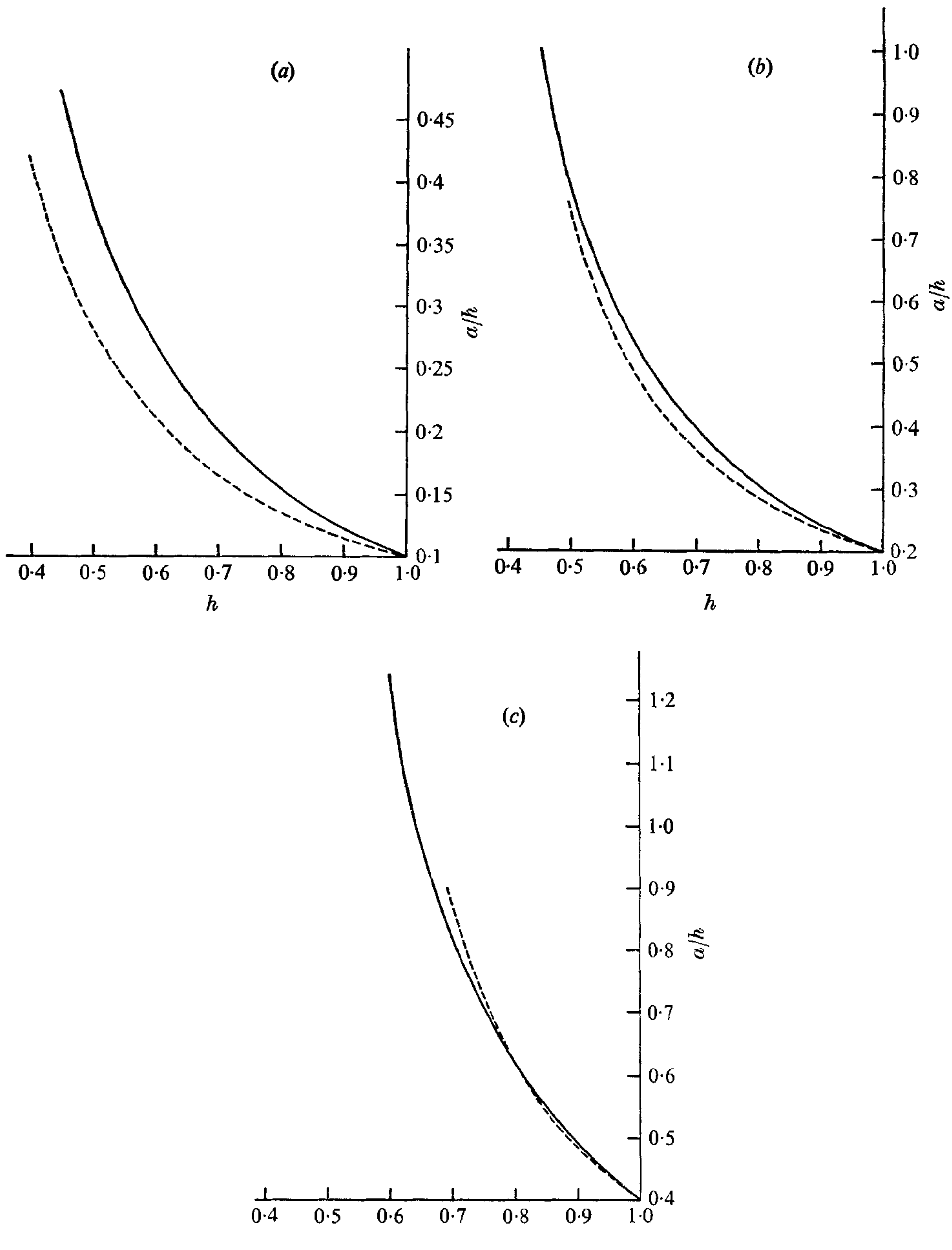

FIGURE 1. Amplitude variation with depth. ——, theoretical results; - - - -, experimental results of Camfield & Street (1969) for beach slope 0·01. (*a*) Initial amplitude $a_0 = 0{\cdot}1$, (*b*) $a_0 = 0{\cdot}2$, (*c*) $a_0 = 0{\cdot}4$.

where V is a function only of c and h (alternatively both V and c are functions of α, defined in (2.28), and h). Equations (4.4) and (4.5) may then be rewritten in the form

$$V_T + c\nabla_{\mathbf{X}}.(\boldsymbol{\nu} V) = 0, \tag{4.7}$$

$$\kappa_T + \boldsymbol{\nu}.\nabla_{\mathbf{X}}(\kappa c) = 0, \tag{4.8}$$

$$\boldsymbol{\nu}_T + c\boldsymbol{\nu}.\nabla_{\mathbf{X}}\boldsymbol{\nu} + \nabla_{\mathbf{X}}c - \boldsymbol{\nu}(\boldsymbol{\nu}.\nabla_{\mathbf{X}}c) = 0. \tag{4.9}$$

Equations (4.7) and (4.9) provide two equations for V and $\boldsymbol{\nu}$ (or c and $\boldsymbol{\nu}$), and (4.8) then determines κ.

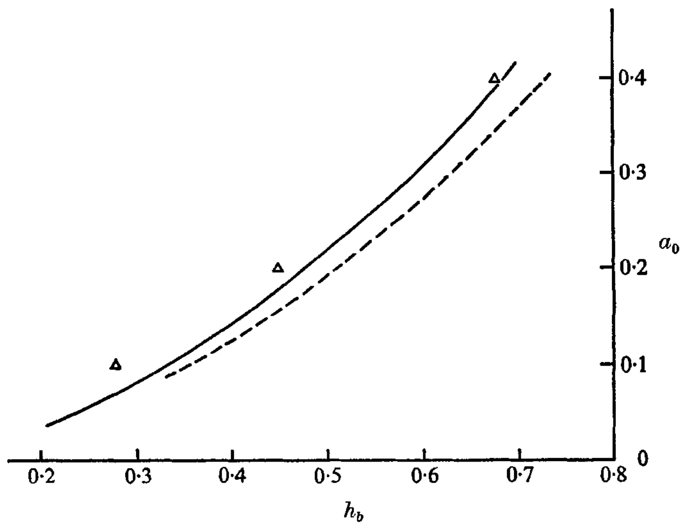

FIGURE 2. The variation of breaking depth with initial amplitude. ——, theoretical results using $\alpha_{\max} = 0{\cdot}86$; - - - - -, theoretical results using $\alpha_{\max} = 0{\cdot}75$; $\triangle$, experimental results of Camfield & Street (1969) for beach slope 0·01.

Since V is the wave average of the energy density with respect to the x scale, (4.7) states that the wave energy is preserved, being carried at a speed c in the direction $\boldsymbol{\nu}$. For small amplitudes α, c is approximately $h^{\frac{1}{2}}$ and (4.9) determines $\boldsymbol{\nu}$ from h; with $\boldsymbol{\nu}$ known, (4.7) may be integrated using familiar procedures. For finite amplitudes (4.7) and (4.9) are coupled and form a system of hyperbolic equations, which may presumably be integrated numerically using well-known techniques. Here, however, we shall merely observe that solving for the characteristics leads to the result that weak discontinuities (in V and $\boldsymbol{\nu}$) will propagate at an angle γ to $\boldsymbol{\nu}$ with the local speed $c\,(\cos\gamma \pm \{(V/c)\,\partial c/\partial V\}^{\frac{1}{2}}\sin\gamma)$.

If h depends only on X, so that the modulation is entirely one-dimensional, then $\boldsymbol{\nu}$ is constant and (4.7) reduces to

$$V_T + cV_X = 0 \tag{4.10}$$

where $c = c(V, h(X))$. The general solution of (4.10) is

$$V = M(T_0),$$

where
$$T_0 = T - \int_0^X ds\{c(M(T_0), h(s))\}^{-1}. \tag{4.11}$$

If the wave is initially uniform (for example, evolving from a region $X \leqslant 0$ where $h = 1$), then $M(T_0)$ is a constant. It may be shown from (2.29) and (2.30) that

$$V = (\tfrac{4}{3}\alpha h^2)^{\frac{3}{2}} \{1+\tfrac{7}{20}\alpha - 0{\cdot}294\alpha^2 + O(\alpha^3)\}^{\frac{3}{2}}, \tag{4.12}$$

where $\alpha = ah^{-1}$. Thus α, and hence a, may be determined as a function of h from

$$\alpha + \tfrac{7}{20}\alpha^2 - 0{\cdot}294\alpha^3 \approx h^{-2}(a_0 + \tfrac{7}{20}a_0^2 - 0{\cdot}294a_0^3), \tag{4.13}$$

where a_0 is the value of a when $h = 1$. For small a_0 this implies that a varies as h^{-1}. For finite a_0 (4.13) was treated as a cubic for α and solved numerically. The results are presented in figure 1, which also contains some experimental results of Camfield & Street (1969) for a beach of constant slope 0·01. They also performed experiments with other slopes and observed that the rate of increase of α as h decreases is made smaller by increasing the beach slope. The most unsatisfactory feature of the present theory would seem to be its inability to predict the variation of α with h_X. If we accept the criterion that the wave will break when $\alpha = \alpha_{\max}$, then substitution into (4.13) determines the breaking depth h_b as a function of a_0. Figure 2 shows that the results of this calculation when $\alpha_{\max} = 0{\cdot}86$ (a theoretical value) and when $\alpha_{\max} = 0{\cdot}75$ (an experimental value obtained by Camfield & Street for small slopes); also shown are some experimental values obtained by Camfield & Street (1969).

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